

A PI DEGREE THEOREM FOR QUANTUM DEFORMATIONS

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1. INTRODUCTION

Let F be an algebraically closed field. We show that if a quantum formal deformation A of a commutative domain A_0 over F is a PI algebra, then A is commutative if $\text{char}(F) = 0$, and has PI degree a power of p if $\text{char}(F) = p > 0$. This implies the same result for filtered deformations (i.e., filtered algebras A such that $\text{gr}(A) = A_0$).

Note that a quantum formal deformation of a commutative domain A_0 may fail to be PI, even for finitely generated A_0 in characteristic p (Example 3.3(2)). However, we don't know if this is possible for filtered deformations. Thus we propose

Question 1.1. Let $\text{char}(F) = p > 0$, and A be a filtered deformation of a commutative finitely generated domain A_0 over F . Must A be a PI algebra? In other words, must the division ring of quotients of A be a central simple algebra?

This question is closely related to the question asked in the introduction to [CEW], which would have affirmative answer if the answer to Question 1.1 is affirmative. We don't know the answer to either of these questions even when A_0 is a polynomial algebra with generators in positive degrees.

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2. DEFORMATIONS OF FIELDS

Let F be an algebraically closed field, and A_0 a field extension of F . Let A be a quantum formal deformation of A_0 over $F[[\hbar]]$, i.e. an $F[[\hbar]]$ -algebra isomorphic to $A_0[[\hbar]]$ as an $F[[\hbar]]$ module and equipped with an isomorphism of algebras $A/(\hbar) \cong A_0$ (for basics and notation on deformations, see [EW], Section 2).

Theorem 2.1. *Suppose that A is a PI algebra of degree d .*

(i) If $\text{char} F = 0$, then $d = 1$ (i.e., A is commutative).

(ii) If $\text{char} F = p > 0$, then d is a power of p .

Proof. Let C be the center of A . It is easy to see that the division algebra of quotients of A is $A[\hbar^{-1}]$ with center $C[\hbar^{-1}]$ (see [EW], Example 2.7). Moreover, by Posner's theorem ([MR], 13.6.5), $A[\hbar^{-1}]$ is a central division algebra over $C[\hbar^{-1}]$ of degree d , so $[A[\hbar^{-1}] : C[\hbar^{-1}]] = d^2$.

Let $C_0 = C/(\hbar)$. It is clear that C_0 is a subfield of A_0 , and C is a (commutative) formal deformation of C_0 .

Lemma 2.2. $[A_0 : C_0] = d^2$.

Proof. Let $a_1^0, \dots, a_m^0 \in A_0$ be linearly independent over C_0 . Let a_1, \dots, a_m be lifts of these elements to A . Then a_1, \dots, a_m are linearly independent over C and hence over $C[\hbar^{-1}]$. Thus $m \leq d^2$. Moreover, if a_1^0, \dots, a_m^0 are a basis of A_0 over C_0 then a_1, \dots, a_m are a free basis of A over C and hence a basis of $A[\hbar^{-1}]$ over $C[\hbar^{-1}]$, so $m = d^2$. \square

Now for every integer $r \geq 0$, let $A_r \subset A_0$ be the field of all $x \in A_0$ which admit a lift to a central element of $A/(\hbar^{r+1})$. Note that $A_r \supset A_{r+1}$, and by Lemma 2.2, this is a finite field extension.

Let us now prove (i). Assume the contrary, i.e. that A is noncommutative. Let r be the largest integer such that $[a, b] \in \hbar^r A$ for all $a, b \in A$. Then we have a nonzero Poisson bracket on A_0 given by $\{a_0, b_0\} = \hbar^{-r}[a, b] \bmod \hbar$, where a, b are any lifts of a_0, b_0 to A . Moreover, by definition $\{, \}$ is bilinear over A_r . Recall that $\{, \}$ is a derivation in each argument, and that any K -linear derivation of a finite extension of a field K of characteristic zero vanishes. Since $[A_0 : A_r] < \infty$, this implies that $\{, \} = 0$, a contradiction. This proves (i).

We now prove (ii).

Lemma 2.3. For large enough r , $A_r = C_0$.

Proof. For nonnegative integers $r \geq s$, let $C_{r,s} \subset A/(\hbar^{s+1})$ be the set of elements liftable to a central element of $A/(\hbar^{r+1})$. It is clear that $C_{s,s}$ is the center of $A/(\hbar^{s+1})$, $C_{r,s} \supset C_{r+1,s}$, and $C_{r,s-1}$ is a quotient of $C_{r,s}$. Also $C_{r,s}$ is a $C_0/(\hbar^{s+1})$ -submodule of $A/(\hbar^{s+1})$. Let $C_{\infty,s}$ be the intersection of $C_{r,s}$ over all r . By Lemma 2.2, $A/(\hbar^{s+1})$ has finite length as a $C_0/(\hbar^{s+1})$ -module, so $C_{\infty,s} = C_{r(s),s}$ for a suitable $r(s)$. This implies that the natural map $C_{\infty,s} \rightarrow C_{\infty,s-1}$ is surjective (as it coincides with the map $C_{r,s} \rightarrow C_{r,s-1}$ for a suitable r). Let $C_{\infty,\infty} = \varprojlim C_{\infty,s} \subset A$.

We claim that any element $a \in C_{\infty,\infty}$ is central in A . Indeed, a projects to $a_s \in C_{\infty,s} \subset C_{s,s}$ which is central in $A/(\hbar^{s+1})$. Hence for any $b \in A$ we have $[a, b] = O(\hbar^{s+1})$. Since this holds for all s , we get that $[a, b] = 0$.

This implies that $C_{\infty, \infty} = C$ (as $C_{\infty, \infty}$ clearly contains C). Hence $C_{\infty, s} = C/(\hbar^{s+1})$ and in particular $C_{\infty, 0} = C_0$. Hence $C_{r, 0} = C_0$ for a large enough r . But by definition $C_{r, 0} = A_r$, which implies the lemma. \square

Lemma 2.4. *For all $r \geq 0$, one has $A_{r+1} \supset A_r^p$.*

Proof. Let $a_0 \in A_r$, and a be its lift to A central modulo \hbar^{r+1} . Let $b \in A$. We have

$$[a^p, b] = \sum_{i=0}^{p-1} a^i [a, b] a^{p-1-i} = p[a, b] a^{p-1} + \sum_{i=0}^{p-1} [a^i, [a, b]] a^{p-1-i} =$$

$$\sum_{i=0}^{p-1} [a^i, [a, b]] a^{p-1-i}$$

(as we are in characteristic p). We have $[a, b] \in \hbar^{r+1}A$, hence $[a^i, [a, b]] \in \hbar^{r+2}A$. Thus $a^p \in A_{r+1}$. \square

Lemma 2.4 implies that A_r is a purely inseparable extension of A_{r+1} . In particular, $[A_r : A_{r+1}]$ is a power of p . Since by Lemma 2.3 $A_r = C_0$ for large r , this implies that $[A_0 : C_0]$, and hence d , is a power of p , as desired. \square

Remark 2.5. Here is another proof of Theorem 2.1 (which deviates from the above proof after Lemma 2.2). By Lemma 2.2, it suffices to show that A_0 is a purely inseparable extension of C_0 (in particular, $A_0 = C_0$ in characteristic zero). To this end, consider the algebra $B := A \otimes_C A^{\text{op}}$. Since $A[\hbar^{-1}]$ is a central division algebra of degree d over $C[\hbar^{-1}]$, we have $B[\hbar^{-1}] \cong \text{Mat}_d(C[\hbar^{-1}])$, hence B does not contain nontrivial central idempotents. Therefore, the same holds for $B_0 := B/(\hbar)$ (otherwise we would have a nontrivial decomposition $B_0 = B'_0 \oplus B''_0$, which would lift to a decomposition $B = B' \oplus B''$, and $1_{B'}$ would be a nontrivial central idempotent in B). But $B_0 = A_0 \otimes_{C_0} A_0$. Hence B_0 has no nontrivial idempotents (i.e., is local). Let $x \in A_0$ be a separable element over C_0 and $K := C_0[x] \subset A_0$. Then $K \otimes_{C_0} K \subset A_0 \otimes_{C_0} A_0$ is reduced and projects onto K , hence contains nontrivial idempotents unless $K = C_0$. Hence $x \in C_0$, and A_0 is purely inseparable over C_0 , as desired.

3. DEFORMATIONS OF DOMAINS

Let us now extend Theorem 2.1 to deformations of domains.

Theorem 3.1. *Theorem 2.1 holds more generally, if A_0 is a domain over F .*

Proof. Following [EW], Subsection 2.2, let $Q(A) = \varprojlim Q(A/(\hbar^{N+1}))$, where $Q(A/(\hbar^{N+1}))$ is the classical quotient ring of $A/(\hbar^{N+1})$.¹ Also, let $Q_*(A) \subset Q(A)[\hbar^{-1}]$ be the quotient division algebra of A (which exists since A is a PI domain). Then $Q_*(A)$ is dense in $Q(A)[\hbar^{-1}]$ in the \hbar -adic topology (although in general $Q_*(A) \neq Q(A)$), and hence satisfies the same polynomial identities as $Q(A)[\hbar^{-1}]$. By Posner's theorem, $Q_*(A)$ is a central division algebra of degree d , hence so is $Q(A)[\hbar^{-1}]$ (as it is a division algebra satisfying the identities of $d \times d$ matrices but not matrices of smaller size). Also, $Q(A)$ is a formal quantum deformation of the quotient field $Q(A_0)$ of A_0 , which is a field extension of F . Thus, Theorem 2.1 applies to $Q(A)$, and the theorem is proved. \square

Corollary 3.2. *Let A be a \mathbb{Z}_+ -filtered deformation of a commutative domain A_0 over F (i.e., $\text{gr}(A) = A_0$). Suppose that A is a PI algebra of degree d .*

- (i) *If $\text{char} F = 0$, then $d = 1$ (i.e., A is commutative).*
- (ii) *If $\text{char} F = p > 0$ then d is a power of p .*

Proof. Let $R(A)$ be the Rees algebra of R and $\widehat{R}(A)$ the completed Rees algebra of A (see e.g. [EW], Subsection 2.1). Then $R(A)$ satisfies the identities of matrices of size $d \times d$ but not smaller (since so does A). Since $R(A)$ is dense in $\widehat{R}(A)$ in the \hbar -adic topology, the same holds for $\widehat{R}(A)$. But $\widehat{R}(A)$ is a formal quantum deformation of A . Thus Theorem 3.1 implies the result. \square

Example 3.3. 1. Suppose $\text{char} F = p > 0$. Let A be the formal n -th Weyl algebra, i.e. the \hbar -adically complete algebra over $F[[\hbar]]$ generated by $x_1, \dots, x_n, y_1, \dots, y_n$ with defining relations

$$[x_i, x_j] = [y_i, y_j] = 0, \quad [y_i, x_j] = \hbar \delta_{ij}.$$

Then A is a formal deformation of $A_0 := F[x_1, \dots, x_n, y_1, \dots, y_n]$, which is the completed Rees algebra of its filtered deformation (the usual Weyl algebra $\mathbf{A}_n(F)$). The center A is $C = F[x_1^p, \dots, x_n^p, y_1^p, \dots, y_n^p][[\hbar]]$, so A is PI of degree $d = p^n$. Note that if we have infinitely many generators x_i, y_i then A is not PI, so the “finitely generated” assumption in Question 1.1 is needed.

2. A formal quantum deformation of a finitely generated commutative domain does not have to be PI, even in characteristic p . E.g., let A be the formal quantum polynomial algebra, i.e. the \hbar -adically complete algebra generated by x, y with relation $yx = (1 + \hbar)xy$. This algebra is a quantum formal deformation of $A_0 := F[x, y]$. It has trivial center and hence is not PI.

¹The characteristic zero assumption of [EW] is not used in these considerations.

Remark 3.4. Here is a direct proof of Corollary 3.2(i), bypassing localizations and formal deformations.

Let C be the center of A and $C_0 = \text{gr}(C)$. We claim that A_0 is algebraic over C_0 . To show this, let $a_0 \in A_0$ be a homogeneous element, and lift it to an element $a \in A$. Since A is PI, by Posner's theorem it is algebraic over C , so there exists a nonzero $P \in C[t]$ such that $P(a) = 0$. Taking the leading terms of this equation gives a nonzero polynomial $P_0 \in C_0[t]$ such that $P_0(a_0) = 0$, as desired.

Now assume that A is noncommutative. Let $\{, \}$ be the nonzero Poisson bracket on A_0 defined in the proof of Theorem 2.1(i). Given $a_0 \in A_0$, the operator $\{a_0, ?\}$ is a derivation of A_0 which vanishes on C_0 . Since A_0 is algebraic over C_0 and $\text{char} F = 0$, this derivation vanishes, i.e., $\{, \} = 0$, a contradiction.

The same argument works for formal deformations (Theorem 3.1 when $\text{char} F = 0$).

Remark 3.5. Let us say that an algebra A is locally PI if any finitely generated subalgebra of A is PI. An example of such an algebra is the Weyl algebra $\mathbf{A}_{\mathcal{I}}(F)$ generated by x_i, y_i , $i \in \mathcal{I}$ for an infinite set \mathcal{I} and $\text{char} F = p > 0$. Corollary 3.2 immediately implies that if A is a locally PI filtered quantization of a commutative domain A_0 over F then A must be commutative if $\text{char}(F) = 0$, and the PI degree of every finitely generated subalgebra of A is a power of p if $\text{char}(F) = p > 0$. Thus, in the special case when A is a connected Hopf algebra equipped with the coradical filtration and $\text{char} F = 0$, we recover [BGZ], Theorem 4.5.

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